Module-1 : Fundamentals of Logic

1.1 Introduction

→ The word logic can be understood as the science
of reasoning and 'logical' means valid reasoning.
If an argument is valid, then every argument
with the same form is also valid. If an argument
is invalid, then every argument with the same form
is also invalid.

→ Mathematical Logic is related to all kinds of reasoning
and it serves as a tool to judge the truthfulness of
any statement.

→ Logic principles play an important role in computer
programming and designs. It is also a foundation in
the development of subjects like 'logical design',
'logic gates', 'artificial intelligence' etc.

1.2 propositions

→ An assertion is a statement

→ A proposition is an assertion which is either true or false but not both.

→ propositions are usually represented by small letters p, q, r, a, b, c, - - - -

→ The truth of a propositions is called the truth value

Truth value	
True →	'1' or 'T'
False →	'0' or 'F'

NOTE:- Any sentence or to any assertion to which you can associate a truth value is called a proposition.

→ Consider the following examples. These are propositions

(a) 4 is a prime number

It is an assertion but it is a false statement. So the truth value associated with 4 is a prime number is false.

(b) $3 + 3 = 6$

It is an assertion and it is a correct statement. So the truth value associated with that is true.

(c) The moon is made of cheese.

It is an assertion and a wrong statement. So the truth value associated with that is false.

→ The following are not propositions

(a) $X + Y > 4$

It is true or false? It depends on what value you are going to give for X and Y. It does not have a unique value, it depends on the value you are going to give for the variables X and Y. X and Y are called individual variables. So you cannot associate a unique truth value to this. This is not a proposition.

(b) $X = 3$

Here you find that you cannot associate a truth value to this. If you give the value 3 to X it will be true, if you give some other value to X, it will not be true. This is not a proposition.

(c) Are you leaving?

This is not an assertion. A proposition should be an assertion. Are you leaving is a question and it is not an assertion. So it is not a proposition.

(d) Buy 4 books

This is not an assertion, it is an order, so it cannot be a proposition.

NOTE:- Whenever something is not an assertion, it is a question or an order it cannot be a proposition. But in special cases there can be assertions which are not propositions.

→ For example, consider this sentence.

$\boxed{\text{This statement is false}}$

↳ It is an assertion, it is not a proposition because you cannot associate a truth value to this. If it is true, it is false. If it is false, it is true. So you cannot associate a true or a false value with this statement. This is called a paradox / a liar paradox.

1.3 Basic Connectives and Truth tables

There are several connectives on the propositions. We present three basic logical connectives on the propositions using simple english words 'and', 'not', 'or'.

'and' → $\wedge$, 'not' → $\sim$, 'or' → $\vee$

→ Compound proposition

The new proposition is obtained by combining the two given propositions using the logical connectives are called as compound proposition.

→ Simple proposition

proposition which do not contains any logical connectives are called simple propositions.

→ Negation($\sim$)/($\neg$)

- A proposition is obtained by inserting the word 'not' in an appropriate place is called the negation of the given proposition.
- The negation of a proposition 'p' is denoted by $\sim p$

Example ① p: 3 is a prime number − 1

$\sim p$: 3 is not a prime number − 0

Truth table

p	$\sim p$
1	0
0	1

② p: 8 is divisible by 3 − 0

$\sim p$: 8 is not divisible by 3 − 1

Truth table

p	$\sim p$
0	1
1	0

→ Conjunction (∧)

- A compound proposition is obtained by inserting the word 'and' between two given propositions is called conjunction of the given proposition.
- The conjunction of p and q is denoted by $p \wedge q$
- The conjunction of p and q are true, only when both p and q are true, in all other cases it is false.

Example:- Consider the proposition

p: John is 6 feet tall

q: There are 4 cows in the barn

p and q: John is 6 feet tall and there are 4 cows in the barn.

Usually we denote true by '1' and false by '0'

Truth table → Truth table

p	q	$p \wedge q$
F	F	F
F	T	F
T	F	F
T	T	T

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

→ Disjunction (∨) / Inclusive disjunction (∨)

- A compound proportion is obtained by inserting the word 'or' between two given propositions is called disjunction of the given proposition.
- The Inclusive disjunction of p and q is denoted by $p \vee q$
- The Inclusive disjunction of p and q are false only when both p and q are false, in all other cases it is true.

Example:- Consider the proposition

p: John is 6 feet tall

q: There are 4 cows in the barn.

p or q : John is 6 feet tall or there are 4 cows in the barn

Truth table

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

→ Exclusive disjunction ($\veebar$)

- The exclusive disjunction of two propositions p and q is denoted by $p \veebar q$ (Read it as either p or q)
- The exclusive disjunction is true only when either p is true (or) q is true but not both

Truth table

p	q	$p \veebar q$
F	F	F
F	T	T
T	F	T
T	T	F

p	q	$p \veebar q$
0	0	0
0	1	1
1	0	1
1	1	0

~~Worked problems~~

~~(1) Write down the truth value of the following compound propositions~~

~~(i) Rajmahal is in Delhi and B~~

Worked problems

1. State which of the following sentences are propositions

(a) A triangle contains three sides - True propositions

(b) $x+2$ is a positive integer - Not a proposition

(c) 5 divides x - Not a proposition

(d) Is $\sqrt{2}$ a rational number? - Not a proposition

(e) 21 is an even number - False proposition

2. Indicate the negation of each of the following propositions.

(a) $2+3=5$ - $2+3 \neq 5$

(b) p: 5 divides 27

$\sim p$: 5 does not divide 27

(c) q: Computer science is a hard subject

$\sim q$: Computer science is not a hard subject

(d) r: Bangalore has pleasant weather.

$\neg r$: Bangalore does not have pleasant weather.

(3) Give the conjunction and disjunction of p and q in the following cases; in each case indicate the truth value.

(i) p: 4 is a perfect square.
q: 27 is a prime number

Sol:- $p \wedge q$: 4 is a perfect square and 27 is a prime number; truth value 0.

$p \vee q$: 4 is a perfect square or 27 is a prime number; truth value 1.

(ii) p : 5 is divisible by 2
q: 7 is a multiple of 5.

Sol:- $p \wedge q$: 5 is divisible by 2 and 7 is a multiple of 5; truth value 0.

$p \vee q$: 5 is divisible by 2 or 7 is a multiple of 5; truth value 0.

(4) Construct truth table for the following

(i) $p \wedge (q \wedge p)$

p	q	$q \wedge p$	$p \wedge (q \wedge p)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

(ii) $\sim(p \vee \sim q)$

Sol:-

p	q	$\sim q$	$p \vee \sim q$	$\sim(p \vee \sim q)$
T	T	F	T	F
T	F	T	T	F
F	T	F	F	T
F	F	T	T	F

(iii) $(p \vee q) \vee \sim p$

p	q	$p \vee q$	$\sim p$	$(p \vee q) \vee \sim p$
T	T	T	F	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	T

(iv) $p \wedge (q \vee r)$ & $p \vee (q \vee r)$

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \vee (q \vee r)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	F	F	T
F	T	T	T	F	T
F	T	F	T	F	T
F	F	T	T	F	T
F	F	F	F	F	F

(v) $\sim p \veebar \sim q$ & $\sim p \vee \sim q$

Sol:-

p	q	$\sim p$	$\sim q$	$\sim p \veebar \sim q$	$\sim p \vee \sim q$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	F	T

Exercise

(1) Construct the truth tables for the following

(a) $p \vee (\neg q)$ (b) $p \wedge (q \vee p)$

(c)

(*) Conditional ($\rightarrow$)

- A compound proposition is obtained by combining two given propositions p and q by using the words "if" before p and "then" before q is called the conditional proposition.
- The conditional "If p, then q" is denoted by $p \rightarrow q$.
- The conditional "If q, then p" is denoted by $q \rightarrow p$.
- The conditional $p \rightarrow q$ is false only when p is true and q is false, in all other cases it is true.

Ex: p: 2 is a prime number

q: 3 is a prime number

r: 6 is a perfect square

s: 9 is a multiple of 6

$p \rightarrow q$: If 2 is a prime number, then 3 is a prime number, then 6 is a perfect square.

$p \rightarrow r$: If 2 is a prime number, then 6 is a perfect square.

$r \rightarrow p$: If 6 is a perfect square, then 2 is a prime number.

$r \rightarrow s$: If 6 is a perfect square, then 9 is a multiple of 6.

Here, $p \rightarrow q$ is true,

$p \rightarrow r$ is false,

$r \rightarrow p$ is true,

$r \rightarrow s$ is true.

Truth table for conditional proposition

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

p	q	p → q
1	1	1
1	0	0
0	1	1
0	0	1

Note:- Let us form the truth table of $q \to p$

q	p	q → p
T	T	T
F	T	T
T	F	F
F	F	T

q	p	q → p
1	1	1
0	1	1
1	0	0
0	0	1

It is important to note that $q \to p$ is not the same as $p \to q$

(*) Bi-Conditional ($\longleftrightarrow$)

- A Compound proposition is obtained by introducing the word 'if and only if' in an appropriate place is called as Bi-Conditional of the given proposition.

(or)

- Let p and q be two propositions. Then the conjunction of the Conditionals $p \rightarrow q$ and $q \rightarrow p$ is called the biconditional of p and q; it is denoted by $p \longleftrightarrow q$.
- The biconditional $p \longleftrightarrow q$ is true only when both p and q have the same truth values otherwise it is false.
- The biconditional of p and q is denoted by

$$p \longleftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)$$

Ex:- p: 2 is a prime number, q: 3 is a prime number
r: 6 is a perfect square, s: 9 is a multiple of 6.

- $p \longleftrightarrow q$: 2 is a prime number if and only if 3 is a prime number. – True
- $p \longleftrightarrow r$: 2 is a prime number if and only if 6 is a perfect square – False
- $r \longleftrightarrow s$: 6 is a perfect square if and only if 9 is a multiple of 6 – True
- $p \longleftrightarrow s$: 2 is a prime number if and only if 9 is a multiple of 6 – False.

Truth table for biconditional propositions

p	q	p→q	q→p	p⟷q
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

p	q	p→q	q→p	p⟷q
1	1	1	1	1
1	0	0	1	0
0	1	1	0	0
0	0	1	1	1

NOTE:- Let p and q be two propositions. Also $p \to q$ is the conditional proposition. Then

(i) proposition $\boxed{q \to p}$ is called its converse

(ii) proposition $\boxed{\sim p \to \sim q}$ is called its inverse.

(iii) proposition $\boxed{\sim q \to \sim p}$ is called its contraposition.

Truth table

p	q	~p	~q	converse q→p	Inverse ~p→~q	contraposition ~q→~p	Conditional p→q
T	T	F	F	T	T	T	T
T	F	F	T	T	T	F	F
F	T	T	F	F	F	T	T
F	F	T	T	T	T	T	T

Note that $p \to q$ and $\sim q \to \sim p$ are the same, they are equivalent

$q \to p$ and $\sim p \to \sim q$ are the same, they are equivalent.

(*) Tautology, Contradiction and contingency

→ **Tautology**:- A compound proposition which is true for all possible truth values of its components is called a tautology.

→ **Contradiction**: A compound proposition which is false for all possible truth values of its components is called a contradiction (absurdity).

→ **Contingency**:- A compound proposition ~~proposition~~ which is either true or false is called a contingency.

Equivalently we can say that, if every entry in the final column of the truth table is

(i) $\boxed{\text{True (T)}}$ then the compound proposition is a $\boxed{\text{Tautology}}$

(ii) $\boxed{\text{False (F)}}$ then the compound proposition is a $\boxed{\text{Contradiction}}$

(iii) $\boxed{\text{Either true (T) or false (F)}}$ then the compound proposition is $\boxed{\text{Contingency}}$

Note:- Contingency is a compound proposition which is neither a tautology nor a contradiction.

(*) Logical Equivalence ($\Longleftrightarrow$)

- Two compound propositions P and Q are said to be logically equivalent if P and Q have the same truth value.
- The notation $P \equiv Q$ or $P \Longleftrightarrow Q$ is used for logical equivalence.

Note:- (i) When the propositions P and Q are not logically equivalent, we write $P \not\Longleftrightarrow Q$

(ii) If the biconditional $P \longleftrightarrow Q$ is a tautology then P and Q are logically equivalent.

Worked Examples

(1) Write down the converse, inverse and contrapositive of the following compound proposition "A person is successful in life if he puts sincere efforts".

Sol:- Let p and q represent the following propositions.

p: A person is successful in life

q: He puts sincere efforts

$p \rightarrow q$ is the given proposition.

(i) Converse is $q \rightarrow p$. i.e., "If a person puts sincere efforts he is successful in life".

(ii) Inverse is $\sim p \rightarrow \sim q$. i.e., "If a person is not successful in life then he is not sincere in his efforts".

(iii) Contrapositive is $\sim q \rightarrow \sim p$. i.e., "If a person has not put sincere efforts then he is not successful in life".

② Define Tautology and contradiction with an example for each.

Sol:- **Tautology**:- A compound proposition which is true for all possible truth values of its components is called a tautology.

- **Contradiction**:- A compound proposition which is false for all possible truth values of its components is called a contradiction.

→ An example for tautology: $(p \wedge q) \rightarrow (p \vee q)$

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \rightarrow (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

→ An example for contradiction: $(p \wedge q) \wedge \sim(p \vee q)$

p	q	$p \wedge q$	$p \vee q$	$\sim(p \vee q)$	$(p \wedge q) \wedge \sim(p \vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

Note:- (i) $p \wedge (\sim p)$ is always contradiction for any proposition "p". (simple or compound)

(ii) $(p \wedge q) \longleftrightarrow (p \vee q)$ serves as an example for contingency.

3. p and q are primitive statements with $p \to q$ is false. Determine the truth values of the following.
(i) $p \wedge q$ (ii) $\sim p \vee q$ (iii) $q \to p$ (iv) $\sim q \to \sim p$

Sol:- Since $p \to q$ false, we have p is true(T) and q is false(F)

(i)

p	q	$p \wedge q$
T	F	F

(ii)

$\sim p$	q	$\sim p \vee q$
F	F	F

(iii)

q	p	$q \to p$
F	T	T

(iv)

$\sim q$	$\sim p$	$\sim q \to \sim p$
T	F	F

4. Verify that $[p \to (q \to r)] \to [(p \to q) \to (p \to r)]$ is a tautology.

Sol:- Construct the associated truth table

p	q	r	$p \to q$ (A)	$q \to r$ (B)	$p \to r$ (C)	$p \to B$ (D)	$A \to C$ (E)	$D \to E$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	F	T	T	T	T	T
F	T	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T
F	F	T	T	T	T	T	T	T
F	T	F	T	F	T	T	T	T
T	F	F	F	T	F	T	T	T

- Observe that all the truth values in the last column of the truth table is true (T).
- Therefore, the given proposition is a tautology.

(5) By constructing truth table show that

$[(p \vee q) \rightarrow r] \Longleftrightarrow [(p \rightarrow r) \wedge (q \rightarrow r)]$

Sol:- The truth table is as follows

p	q	r	$p \vee q$	$p \rightarrow r$	$q \rightarrow r$	$(p \vee q) \rightarrow r$ (A)	$(p \rightarrow r) \wedge (q \rightarrow r)$ (B)	$A \longleftrightarrow B$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	T	T	T	T	T	T
F	T	T	T	T	T	T	T	T
F	F	F	F	T	T	T	T	T
F	F	T	F	T	T	T	T	T
F	T	F	T	T	F	F	F	T
T	F	F	T	F	T	F	F	T

- The left and right portion of the given proposition are respectively denoted by A and B. The truth values of these are same.
- Further $A \longleftrightarrow B$ is a tautology
- Thus we conclude that $A \equiv B$ or $A \Longleftrightarrow B$

Exercise

(1) Given that p, q, r are propositions having truth values 0, 0, 1 respectively. Find the truth value of the following propositions.

(i) $p \rightarrow (q \wedge r)$

(ii) $(p \vee q) \vee r$

(iii) $(p \wedge q) \rightarrow r$

(iv) $p \rightarrow [q \rightarrow (\neg r)]$

(2) Construct truth table for each of the following.

(i) $\sim(p \vee \sim q) \rightarrow \sim p$

(ii) $(p \rightarrow q) \rightarrow (q \rightarrow p)$

(iii) $[q \wedge (p \rightarrow q)] \rightarrow p$

(iv) $q \leftrightarrow [(\sim p) \vee (\sim q)]$

(3) Define tautology. Determine whether the following compound statement is a tautology or not.

$\{(p \vee q) \rightarrow r\} \leftrightarrow \{\neg r \rightarrow \neg(p \vee q)\}$

(4) show that $\{(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)\} \rightarrow r$ is a tautology by constructing the truth table.

(5) Use truth tables to establish the following logical equivalences

(i) $\neg(p \vee q) \Longleftrightarrow \neg p \wedge \neg q$

(ii) $(p \rightarrow q) \Longleftrightarrow \neg p \vee q$

(iii) $p \underline{\vee} q \Longleftrightarrow (p \vee q) \wedge (\neg p \vee \neg q)$

(iv) $p \leftrightarrow q \Longleftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$

(*) Laws of Logic

Involved with propositions P, Q, R and logical operators $\wedge, \vee, \sim(\neg), \rightarrow$

① Commutative laws $P \vee Q \Longleftrightarrow Q \vee P$

$P \wedge Q \Longleftrightarrow Q \wedge P$

② Associative laws $P \vee (Q \vee R) \Longleftrightarrow (P \vee Q) \vee R$

$P \wedge (Q \wedge R) \Longleftrightarrow (P \wedge Q) \wedge R$

③ Distributive laws $P \vee (Q \wedge R) \Longleftrightarrow (P \vee Q) \wedge (P \vee R)$

$P \wedge (Q \vee R) \Longleftrightarrow (P \wedge Q) \vee (P \wedge R)$

④ De-Morgan's laws $\sim(P \vee Q) \Longleftrightarrow \sim P \wedge \sim Q$

$\sim(P \wedge Q) \Longleftrightarrow \sim P \vee \sim Q$

⑤ Absorption laws $P \vee (P \wedge Q) \Longleftrightarrow P$

$P \wedge (P \vee Q) \Longleftrightarrow P$

⑥ Double negation law $\sim(\sim P) \Longleftrightarrow P$

⑦ Idempotent laws $P \vee P \Longleftrightarrow P$; $P \wedge P \Longleftrightarrow P$

⑧ Negation of Conditional law $\sim(P \rightarrow Q) \Longleftrightarrow P \wedge (\sim Q)$

⑨ Law for the Conditional $P \rightarrow Q \Longleftrightarrow \sim P \vee Q$

· Involved with a proposition p and truth values ~~~~ True(T) or tautology(T_t) and False(F) or Contradiction(F_c) associated with logical operators ~, ∨ and ∧

(10) Identity laws $p \wedge T_t \Longleftrightarrow p$; $p \vee F_c \Longleftrightarrow p$

(11) Domination laws $p \vee T_t \Longleftrightarrow T_t$; $p \wedge F_c \Longleftrightarrow F_c$

(12) Negation laws (or) Inverse laws · $p \vee (\sim p) \Longleftrightarrow T_t$; $p \wedge (\sim p) \Longleftrightarrow F_c$

(*) Duality of proposition

- The word dual in a general perspective is 'consisting two parts'.
- Two propositions p and q involving basic connectives '∨', '∧', '~' are said to be duals of each other, if either of them can be obtained from the other on replacement of '∨' by '∧' and '∧' by '∨'. ('~' remains unchanged).
- If a propositions is involved with truth values T (or tautology T_t/T_0) and 'F' (or contradiction F_c/F_0) the dual of the given proposition is obtained on replacement of 'T' by 'F' and 'F' by 'T'. (or T_t by F_c and vice versa).
- Dual of a proposition p is denoted by p^D.

Note:- ① If $P \Leftrightarrow Q$ then $P^D \Leftrightarrow Q^D$ and vice versa.

② $(P^D)^D \Leftrightarrow P$

③ Important results

(i) $p \underline{\vee} q \Leftrightarrow (p \vee q) \wedge (\sim p \vee \sim q)$

(ii) $p \rightarrow q \Leftrightarrow \sim p \vee q$

(iii) $p \leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\sim p \wedge \sim q)$

Examples

	Given proposition(p)	Dual of the given proposition(p^D)
①	$(p \vee q) \wedge r$	$(p \wedge q) \vee r$
②	$\sim(p \wedge q)$	$\sim(p \vee q)$
③	$(p \vee q) \wedge (\sim p \wedge q)$	$(p \wedge q) \vee (\sim p \vee q)$
④	$p \underline{\vee} q \Leftrightarrow [(p \vee q) \wedge (\sim p \vee \sim q)]$	$(p \wedge q) \vee (\sim p \wedge \sim q)$
⑤	$p \rightarrow q \Leftrightarrow \sim p \vee q$	$\sim p \wedge q$
⑥	$p \leftrightarrow q \Leftrightarrow [(p \wedge q) \vee (\sim p \wedge \sim q)]$	$(p \vee q) \wedge (\sim p \vee \sim q)$

Note:- Dual of $p \underline{\vee} q$ is logically equivalence to $p \leftrightarrow q$ and vice versa.

Worked problems

① Prove the following logical equivalence by using
(i) laws of logic (ii) truth tables.

ⓐ $p \vee [p \wedge (p \vee q)] \Leftrightarrow p$ ⓑ $[(p \vee q) \wedge (p \vee \sim q)] \vee q \Leftrightarrow p \vee q$

Sol:- (i) laws of logic

ⓐ $LHS = p \vee [p \wedge (p \vee q)]$

$LHS \Leftrightarrow p \vee p$ (Absorption law)

$LHS \Leftrightarrow p$ (Idempotent law)

Thus $LHS \Leftrightarrow RHS$

ⓑ $LHS = [(p \vee q) \wedge (p \vee \sim q)] \vee q$

$LHS \Leftrightarrow [p \vee (q \wedge \sim q)] \vee q$ (Distributive law RHS to LHS)

$LHS \Leftrightarrow [p \vee F] \vee q$ (Inverse law)

$LHS \Leftrightarrow [p \vee q] = RHS$ (Identity law)

Thus $LHS \Leftrightarrow RHS$

(ii) By using truth table

p	q	$\sim q$	$p \vee q$	$p \vee \sim q$	$p \wedge (p \vee q)$	$p \vee [p \wedge (p \vee q)]$ (A)	$p \vee p$ (B)	$(p \vee q) \wedge (p \vee \sim q)$ (C)	$C \vee q$ (D)
T	T	F	T	T	T	T	T	T	T
T	F	T	T	T	T	T	T	T	T
F	T	F	T	F	F	F	F	F	T
F	F	T	T	T	F	F	F	F	F

Here, $A \Leftrightarrow p$. Thus (a) is proved
$D \Leftrightarrow p \vee q$. Thus (b) is proved

(2) Using the laws of logic, prove the following equivalence

$$[(\neg p \vee \neg q) \wedge (F_0 \vee p) \wedge p] \Longleftrightarrow p \wedge \neg q.$$

Sol:- $LHS = [(\neg p \vee \neg q) \wedge (F_0 \vee p) \wedge p]$

$LHS \Longleftrightarrow [(\neg p \vee \neg q) \wedge (p \wedge p)]$ (by identity law)

$LHS \Longleftrightarrow [(\neg p \vee \neg q) \wedge p]$ (by idempotent law)

$LHS \Longleftrightarrow [p \wedge (\neg p \vee \neg q)]$ (by commutative law)

$LHS \Longleftrightarrow [(p \wedge \neg p) \vee (p \wedge \neg q)]$ (by distributive law)

$LHS \Longleftrightarrow [F_0 \vee (p \wedge \neg q)]$ (by Inverse law)

$LHS \Longleftrightarrow p \wedge \neg q = RHS$ (by Identity law)

$\therefore LHS \Longleftrightarrow RHS$

Hence proved

3 prove the following using the laws of logic

$[\neg p \wedge (\neg q \wedge r)] \vee [(q \wedge r) \vee (p \wedge r)] \Leftrightarrow r.$

Sol:- Let us consider.

$LHS \Leftrightarrow [\neg p \wedge (\neg q \wedge r)] \vee [(q \wedge r) \vee (p \wedge r)]$

$LHS \Leftrightarrow [(\neg p \wedge \neg q) \wedge r] \vee [(r \wedge q) \vee (r \wedge p)]$ {by associative and commutative laws}

$LHS \Leftrightarrow [\neg (p \vee q) \wedge r] \vee [r \wedge (q \vee p)]$ {by De-Morgan's and Distribution laws}

$LHS \Leftrightarrow [\neg (p \vee q) \wedge r] \vee [(q \vee p) \wedge r]$ {by commutative law}

$LHS \Leftrightarrow [\neg (p \vee q) \wedge r] \vee [(p \vee q) \wedge r]$

$LHS \Leftrightarrow [\neg (p \vee q) \vee (p \vee q)] \wedge r$ {by distributive law}

$LHS \Leftrightarrow T_t \wedge r$ {by Inverse law}

$LHS \Leftrightarrow r$ {by Identity law}

4 Prove that $[(\neg p \vee q) \wedge \{p \wedge (p \wedge q)\}] \Leftrightarrow p \wedge q$. Hence deduce that $[(\neg p \wedge q) \vee \{p \vee (p \vee q)\}] \Leftrightarrow p \vee q$

Sol:- Consider,

$LHS = [(\neg p \vee q) \wedge \{p \wedge (p \wedge q)\}]$

$LHS \Leftrightarrow [(\neg p \vee q) \wedge \{(p \wedge p) \wedge q\}]$ (by associative law)

$LHS \Leftrightarrow [(\neg p \vee q) \wedge \{p \wedge q\}]$ (by Idempotent law)

$LHS \Leftrightarrow [\{\neg p \wedge (p \wedge q)\} \vee \{q \wedge (p \wedge q)\}]$ (by distributive law)

$LHS \Leftrightarrow [\{(\neg p \wedge p) \wedge q\} \vee \{(q \wedge p) \wedge q\}]$ (by associate law)

$LHS \Leftrightarrow [(F_c \wedge q) \vee (q \wedge (q \wedge p))]$ (by Inverse and commutative law)

$LHS \Leftrightarrow F \vee [(q \wedge q) \wedge p)]$ (by associative law

$LHS \Leftrightarrow F \vee (q \wedge p)$ (by Idempotent law)

$LHS \Leftrightarrow q \wedge p$ (by Identity law)

$LHS \Leftrightarrow p \wedge q = RHS$ (by commutative)

Thus $LHS \Leftrightarrow RHS$

Now, take Dual of $P = P^D = [(\neg p \wedge q) \vee \{p \vee (p \vee q)\}] \Leftrightarrow p \vee q$, which is true by principle of duality

Hence we have deduced the required result.

5. verify the principle of duality for the following logical equivalence.

$$[\neg(p\wedge q)\rightarrow\{\neg p\vee(\neg p\vee q)\}] \Leftrightarrow \neg p\vee q.$$

Sol:-

Consider,

$LHS = [\neg(p\wedge q)\rightarrow\{\neg p\vee(\neg p\vee q)\}]$

$LHS \Leftrightarrow \neg\{\neg(p\wedge q)\}\vee\{\neg p\vee(\neg p\vee q)\}$ (using result $p\rightarrow q \Leftrightarrow \sim p\vee q$)

$LHS \Leftrightarrow (p\wedge q)\vee\{\neg p\vee(\neg p\vee q)\}$ (by law of double negation)

Let us take

$R = [(p\wedge q)\vee\{\neg p\vee(\neg p\vee q)\}] \Leftrightarrow \neg p\vee q.$

Now we have to verify the principle of duality of R

i.e., $R^D = [(p\vee q)\wedge\{\neg p\wedge(\neg p\wedge q)\}] \Leftrightarrow \neg p\wedge q$

Consider,

$LHS = [(p\vee q)\wedge\{\neg p\wedge(\neg p\wedge q)\}]$

$\Leftrightarrow [(p\vee q)\wedge\{(\neg p\wedge\neg p)\wedge q\}]$ (by associative law)

$\Leftrightarrow [(p\vee q)\wedge(\neg p\wedge q)]$ (by Idempotent law)

$\Leftrightarrow [\{p\wedge(\neg p\wedge q)\}\vee\{q\wedge(\neg p\wedge q)\}]$ (by distributive law)

$\Leftrightarrow [\{(p\wedge\neg p)\wedge q\}\vee\{q\wedge(q\wedge\neg p)\}]$ (by associative and commutative law)

$\Leftrightarrow \{(F_c\wedge q)\vee\{(q\wedge q)\wedge\neg p\}$ (by Inverse and associative)

$\Leftrightarrow F_c\vee(q\wedge\neg p)$ (by domination and Idempotent)

$\Leftrightarrow q\wedge\neg p$ (by Identity law)

$\Leftrightarrow \neg p\wedge q = RHS$ (by commutative)

$LHS \Leftrightarrow RHS$, Hence verified duality of proposition.

Exercise

(1) prove the following logical equivalence by using the laws of logic. Also write down their duals.

(i) $(p \rightarrow q) \wedge [\sim q \wedge (r \vee \sim q)] \Leftrightarrow \sim(q \vee p)$

(ii) $[(\neg p \vee \neg q) \rightarrow (p \wedge q \wedge r)] \Leftrightarrow p \wedge q$

(iii) $p \rightarrow (q \rightarrow r) \Leftrightarrow (p \wedge q) \rightarrow r$

(iv) $[\neg p \wedge (\neg q \wedge r)] \vee [(q \wedge r) \vee (p \wedge r)] \Leftrightarrow r$

(*) Definitions and the Proofs of theorems:

- **Definition** :- It can be understood as a statement of the meaning of a word.

- **Theorem** :- It can be understand as "A scientific or mathematical proposition that can be proved by valid reasoning".

- The propositions that commonly appear in mathematical discussions are conditionals of the form $p \rightarrow q$, where p and q are simple (or) compound proposition and which may involves quantifiers also.

(*) proofs of theorems

- **Direct proof** :- The direct proof of a conditional $p \rightarrow q$ has the following steps

① **Hypothesis** :- Assume that p is true

② **Analysis** :- Starting with the hypothesis and using the rules (or) laws of logic and other known facts, show that q is true (T).

③ **Conclusion** :- $p \rightarrow q$ is true.

Indirect proof

① **Hypothesis**:- Assume that $\sim q$ is true which is equivalent to, p is true (T) and q is false (F).

② **Analysis**:- Starting with the hypothesis and using the rules (or) laws of logic and other known facts, show that $\sim p$ is true (T).

③ **Conclusion**:- $\sim q \longrightarrow \sim p$ (Contrapositive) is true.
Recalling that $p \longrightarrow q \Longleftrightarrow \sim q \longrightarrow \sim p$,
we conclude that $p \longrightarrow q$ is true (T).

(*) Proof by contradiction (Reduction and absurdum)

① **Hypothesis**:- Assume that $p \longrightarrow q$ is false.
i.e., p is true and q is false

② **Analysis**:- Take q is false, using the rules (or) Laws of logic and other known facts, show that p is false.
Hence our assumption is wrong.

③ **Conclusion**:- Hence $p \longrightarrow q$ is true (T).

- Dis-proof by Contradiction

(1) Hypothesis :- Assume that $p \rightarrow q$ is true
i.e., p is true and q is true

(2) Analysis :- Take p is true by using rules, laws or any other known facts to show that q is false. Hence our assumption is wrong.

(3) Conclusion :- $p \rightarrow q$ is false.

- Worked problems

(1) Give (i) A direct proof (ii) An indirect proof (iii) A proof by contradiction for the following statement.
"If n is an odd integer, then $(n+9)$ is an even integer".

Sol:- We first form the propositions
p: n is an odd integer
q: $(n+9)$ is an even integer.

(i) Direct proof

Hypothesis :- Assume that p is true
i.e., n is an odd integer $\Rightarrow n = 2x+1,\ x \in Z$

Worked problems

1. Give (i) A direct proof (ii) An indirect proof (iii) A proof by contradiction for the following statement.

"If n is an odd integer, then $(n+9)$ is an even integer".

Sol:- We first form the propositions,

p: n is an odd integer

q: $(n+9)$ is an even integer

(i) Direct proof:

- Hypothesis:- Assume that p is true i.e., n is an odd integer $\Rightarrow n = 2x+1, x \in Z$.
- Analysis:- Consider, $q: (n+9) = (2x+1)+9 = (2x+10) = 2(x+5)$

  i.e., $q: (n+9) = 2m$, $m = (x+5) \in Z$

  $(n+9) = 2m \Rightarrow (n+9)$ is an even integer
- Conclusion:- ~~$(n+9) = 2m$ implies that $(n+9)$~~

  Thus, if n is odd, then $(n+9)$ is an even integer

  i.e., $p \to q$ is true

(ii) Indirect proof

Hypothesis :- Assume that $\sim q$ is true (or) p is true and q is false

i.e., $\sim q$: $(n+9)$ is an odd integer is true

Analysis

$\Rightarrow (n+9) = (2x+1)$

$\Rightarrow n = 2x+1-9 = 2x-8 = 2(x-4)$

$\Rightarrow n = 2m \,, \; m = x-4 \in Z$

i.e., $n = 2m$ which implies that n is even, this is equivalent to $\sim p$ being true.

Conclusion :-

- Therefore, $\sim q \longrightarrow \sim p$ is true and we have $\sim q \longrightarrow \sim p \Longleftrightarrow p \longrightarrow q$, hence $p \longrightarrow q$ is true.
- Thus, if n is odd integer, then $(n+9)$ is an even integer.

(iii) Proof by Contradiction

Hypothesis :- Assume that $p \longrightarrow q$ is false

i.e., p is true and q is false.

Analysis :-

That is, p : n is an odd integer

q : $(n+9)$ is an odd integer

$\therefore (n+9) = (2x+1) \Rightarrow n = 2x-8 = 2(x-4) = 2m, \; m = x-4 \in Z$

$\Rightarrow n = 2m$ is an even integer

Hence our assumption ~~[illegible]~~ $p \longrightarrow q$ false is wrong.

Conclusion:

Thus, if n is odd then $(n+q)$ is even

(2) Give direct and indirect proof of following statement "product of two odd integers is an odd integer".

Sol: Given, "If m is odd and n is odd then mn is odd".

- Let us form the propositions

p: m is odd

q: n is odd.

r: mn is odd

- The symbolic form is as follows.

$$(p \wedge q) \longrightarrow r$$

Direct proof

- Hypothesis :- Assume that $(p \wedge q)$ is true

$\Rightarrow p$ is true and q is true

- Analysis:- Let m is odd and n is odd

$\Rightarrow m = 2x+1$ and $n = 2y+1$, $x, y \in Z$

$$\Rightarrow mn = (2x+1)(2y+1)$$
$$= 4xy + 2x + 2y + 1$$
$$= 2(2xy + x + y) + 1$$
$$= 2K + 1, \text{ where } K = 2xy + x + y \in Z$$

$\Rightarrow mn = 2K+1$ is odd

Conclusion:- Thus, if m and n are odd then mn is odd integer.

i.e., $(p \wedge q) \rightarrow r$ is true.

Indirect proof.

Hypothesis:- Assume that $\neg r$ is true ~~[illegible]~~

mn is not odd (or)

i.e., $\neg r$: mn is not odd (or)

$\Rightarrow \neg r$: mn is an even integer. is true.

Analysis:- WKT $(p \wedge q) \rightarrow r \Leftrightarrow \neg r \rightarrow \neg(p \wedge q)$ — (1)

We need to show that $\neg(p \wedge q)$ is true

Let, $\neg p$: m is an even integer

$\neg q$: n is an even integer

$\therefore \neg p \vee \neg q$: m is even or n is even is true

$\Rightarrow \neg(p \wedge q)$ is true

$\therefore \neg r \rightarrow \neg(p \wedge q)$ is true

Hence, $(p \wedge q) \rightarrow r$, is true by using (1)

(3) prove that for all integers 'K' and 'l', if K and l both odd, then K+l is even and Kl is odd by direct proof.

Sol: Given, "if K and l both odd, then K+l is even and Kl is odd", for all integers 'K' and 'l'

- Let us form the propositions
  - p: K and l both odd integers
  - q: K+l is an even integers
  - r: Kl is odd integer
- The symbolic form of the statement is as follows

$$p \longrightarrow (q \wedge r)$$

- Direct proof
- Assume that p is true
  i.e. p: K and l both odd integers is true
- We have show that $(q \wedge r)$ is true
- Let $K = 2m+1$ and $l = 2n+1$, $m, n \in \mathbb{Z}$ (integer)

$\Rightarrow q: K+l = (2m+1)+(2n+1)$

$\Rightarrow q: K+l = 2m+2n+2 = 2(m+n+1) = 2x$, where $x = m+n+1 \in \mathbb{Z}$

$\Rightarrow q: K+l = 2x$ is an even integer which is true — (1)

Consider,

$r: Kl = (2m+1)(2n+1) = 4nm + 2m + 2n + 1 = 2(2nm + m + n) + 1$

$\Rightarrow r: Kl = 2y+1$ is an odd integer,

where $y = (2nm + m + n) \in \mathbb{Z}$

$\Rightarrow r: Kl = (2y+1)$ is an odd is true — 2

Now, the Conjunction of q and r is ~~as follows~~ true

i.e., $(q \wedge r)$ is true

Therefore, $p \longrightarrow (q \wedge r)$ is true

(*) Logical Implication - Rules of Inference

- The Compound proposition by prefixing 'if' to p and Prefixing 'then' to q, represents $p \rightarrow q$, read as p implies q. Many times the compound proposition $p \rightarrow q$ apparently appears meaningless/illogical implication when the propositions p and q are not at all related.
- Observe the following two independent propositions.

  p: Delhi is the Capital of India

  q: Chennai is a coastal city.

The Compound proposition $p \rightarrow q$ is, "If Delhi is the Capital of India then Chennai is a coastal City".
This apparently is a meaningless/illogical implication though p and q are independently true proposition

Note:-

- If p and q are any two propositions Such that $p \rightarrow q$ is a tautology, then p logically implies q. (The Compound proposition is meaningful).
- This is denoted by $p \Rightarrow q$

(*) Argument

- Let $p_1, p_2, p_3, ---, p_n$ be a set of propositions. Further let q also be a proposition. Then the compound proposition, $(p_1 \wedge p_2 \wedge p_3 \wedge --- \wedge p_n) \rightarrow q$ is called an argument where $p_1, p_2, p_3, ---, p_n$ are referred to as premises of the argument and q as the conclusion of the argument.
- It is represented in the form of tabular form

i.e,

$$\begin{array}{c} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_n \\ \hline \therefore q \\ \hline \end{array}$$

- The preceding argument is said to be valid if whenever each of the premises $p_1, p_2, ---, p_n$ is true, then the conclusion q is likewise true.
- In otherwords, the argument $(p_1 \wedge p_2 \wedge p_3 \wedge --- \wedge p_n) \rightarrow q$ is valid when

$$(p_1 \wedge p_2 \wedge p_3 \wedge ---- \wedge p_n) \Rightarrow q$$

- For finding the validity of Argument use the Rules of Logic and these rules are called the Rules of Inference.

① Rule of Conjuctive simplification

For any two propositions p and q, if $p \wedge q$ is true then p is true

i.e., $p \wedge q \Rightarrow p$

② Rule of Disjunctive Amplification

For any two propositions p and q, if p is true then $p \vee q$ is true

i.e., $p \Rightarrow p \vee q$

③ Rule of syllogism

- For any three propositions p, q, r if $p \rightarrow q$ is true and $q \rightarrow r$ is true then $p \rightarrow r$ is true

i.e., $\{(p \rightarrow q) \wedge (q \rightarrow r)\} \Rightarrow p \rightarrow r$

- In tabular form

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \\ \hline \end{array}$$

④ Modus pones (Affirm)

This rule states that if p is true and $p \rightarrow q$ is true then q is true

i.e., $\{p \wedge (p \rightarrow q)\} \Rightarrow q$

- In tabular form

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \\ \hline \end{array}$$

5 Modus Tollens (Deny)

This rule states that if $p \rightarrow q$ is true and q is false, then p is false

i.e., $\{(p \rightarrow q) \wedge \sim q\} \Rightarrow \sim p$

In tabular form,

$$\begin{array}{l} p \rightarrow q \\ \sim q \\ \hline \therefore \sim p \\ \hline \end{array}$$

6 Rule of Disjunctive syllogism

This rule states that if $p \vee q$ is true and p is false, then q is true

i.e., $\{(p \vee q) \wedge \sim p\} \Rightarrow q$

In tabular form,

$$\begin{array}{l} p \vee q \\ \sim p \\ \hline \therefore q \\ \hline \end{array}$$

Worked problems

(*) Test the validity of the following arguments

(1) Andrea can program in C++ and she can program in Java. Therefore Andrea can program in C++

Sol:- Write down the premises and the associated conclusion as follows.

Let, p: Andrea can program in C++

q: Andrea can program in Java

The argument is as follows

$$\frac{p \wedge q}{\therefore\ p}$$

i.e., $(p \wedge q) \Longrightarrow p$ (Rule of conjunctive simplification)

Thus we conclude that the argument is valid

(2) If Ram gets distinction in exam, his father will get him a bike.

Ram achieves distinction

$\therefore$ Ram gets bike

Sol:- Let p: Ram gets distinction in exam

q: Ram's father will get him a bike

The argument is as follows

$$\begin{array}{l} p \longrightarrow q \\ \underline{\quad p \quad} \\ \therefore\ q \end{array}$$

i.e., $[(p \rightarrow q) \wedge p] \Longrightarrow q$ (Rule of modus ponens)

Thus we conclude that the argument is valid.

3 Write the following in symbolic form and establish if the argument is valid. If 'A' gets supervisor's position and works hard, then he will get a raise. If he gets a raise then he will buy a new car. He has not bought a new car. Therefore 'A' did not get the supervisor's position or he did not work hard.

Sol:- Let p: 'A' gets supervisor's position

q: 'A' works hard

r: 'A' gets a raise

s: 'A' buys a new car.

The argument in the symbolic form is as follows

$$(p \wedge q) \longrightarrow r$$
$$r \longrightarrow s$$
$$\sim s$$
$$\overline{\therefore \sim p \vee \sim q} \quad \text{———} \quad (1)$$

Denoting $t = p \wedge q$, we have $\{(t \to r) \wedge (r \to s)\} \Rightarrow t \to s$ by rule of syllogism

Now we have the argument as follows

$$t \to s$$
$$\sim s$$
$$\overline{\therefore \sim p \vee \sim q}$$

Further, $(t \to s) \wedge (\sim s) \Rightarrow \sim t$, by the rule of modus pones

Now, $\sim t = \sim(p \wedge q) \Rightarrow \sim p \vee \sim q$ (De-Morgen's law) which is same as in (1)

Thus we conclude that the argument is valid

(4) Test for validity of following argument.

- If Ravi goes out with friends, he will not study
- If Ravi do not study, his father become angry.
- His father is not angry

$\therefore$ Ravi has not gone out with friends

Sol:- Let p: Ravi goes out with friends

q: Ravi will study

r: Ravi's father becomes angry.

Given argument is as follows

$$\begin{array}{l} p \rightarrow \neg q \\ \neg q \rightarrow r \\ \quad \neg r \\ \hline \therefore \quad \neg p \end{array}$$

$$\Rightarrow \quad \begin{array}{l} p \rightarrow r \\ \quad \neg r \\ \hline \therefore \neg p \end{array} \quad \{\text{Rule of syllogism}\}$$

This is a valid argument in view of Modus Tollens (Deny).

5) Check whether the following is a valid argument.

- If I study, then I will not fail in the examination
- If I don't watch TV in the evening, I will study
- I failed in the examination

∴ I must watch TV in the evening.

Sol:- Let us first form the premises as follows

p : I study

q : I will fail in the examination

r : I watch TV in the evening

The argument is as follows

$$\begin{array}{l} p \longrightarrow \sim q \\ \sim r \longrightarrow p \\ \quad q \\ \hline \therefore \quad r \end{array} \quad \text{———— (1)}$$

Consider, $p \longrightarrow \sim q \Rightarrow \sim(\sim q) \longrightarrow \sim p$

i.e., $p \longrightarrow \sim q \Rightarrow q \longrightarrow \sim p$ (Contrapositive)

Also, $\sim r \longrightarrow p \Rightarrow \sim p \longrightarrow \sim(\sim r)$

i.e., $\sim r \longrightarrow p \Rightarrow \sim p \longrightarrow r$ (contrapositive)

Now the argument is as follows

$$\begin{array}{l} q \longrightarrow \sim p \\ \sim p \longrightarrow r \\ \quad q \\ \hline \therefore \quad r \end{array}$$

Further, $(q \to \sim p) \wedge (\sim p \to r) \Rightarrow q \to r$ (Rule of syllogism)

Also, $(q \to r) \wedge q \Rightarrow r$ (Rule of modus pones)

$\quad$ — (2)

Comparing (2) with (1), the argument is valid.

(6) Test the validity of the following arguments.

(i) $(\neg p \vee q) \to r$

$r \to (s \vee t)$

$\neg s \wedge \neg u$

$\neg u \to \neg t$

$\therefore p$

Sol:- Consider,

$[(\neg p \vee q) \to r] \wedge [r \to (s \vee t)] \wedge [\neg s \wedge \neg u] \wedge [\neg u \to \neg t]$

$\Leftrightarrow [(\neg p \vee q) \to (s \vee t)] \wedge \neg s \wedge [\neg u \wedge (\neg u \to \neg t)]$

(by Rule of syllogism and associative law)

$\Leftrightarrow [(\neg p \vee q) \to (s \vee t)] \wedge [\neg s \wedge \neg t]$ (by Modus Pones Rule)

$\Leftrightarrow [(\neg p \vee q) \to (s \vee t)] \wedge [\neg (s \vee t)]$ (by De Morgan law)

$\Leftrightarrow \neg (\neg p \vee q)$ (by Modus Tollens Rule)

$\Leftrightarrow p \wedge \neg q$

$\Rightarrow p$ (by the Rule of conjunctive simplification)

(ii) $p \to r$

$r \to s$

$t \vee \neg s$

$\neg t \vee u$

$\neg u$

$\therefore \neg p$

Sol:- Consider,

$(p \to r) \wedge (r \to s) \wedge (t \vee \neg s) \wedge (\neg t \vee u) \wedge (\neg u)$

$\Rightarrow (p \to s) \wedge (t \vee \neg s) \wedge (\neg t \vee u) \wedge (\neg u)$ {by Rule of syllogism}

$\Rightarrow (p \to s) \wedge (s \to t) \wedge (t \to u) \wedge (\neg u)$ {using the result $p \to q \Longleftrightarrow \sim p \vee q$}

$\Rightarrow (p \to t) \wedge (t \to u) \wedge (\neg u)$ {by Rule of Syllogism}

$\Rightarrow (p \to u) \wedge (\neg u)$ {by Rule of Syllogism}

$\Rightarrow \neg p$ {by Rule of Modus tollens (Deny)}

Hence the given argument is valid.

(*) Quantifiers – Introduction

We have earlier discussed concepts involved with an assertive statement (Proposition) which is either true or false but not both. We discuss the same concepts relating to statements involving variables that turn out as propositions based on the value(s) of the variables. We also discuss quantifiers which is due to the quantity of values of the variables in the statement. Finally we discuss proofs of some theorems involved with integers and study methods of proof and disproof.

→ Standard Alphabets used for the Commonly involved Sets.

$\mathbb{N}$: Set of all natural numbers

$\mathbb{Z}$: Set of all integers

$\mathbb{Q}$: Set of all rational numbers

$\mathbb{R}$: Set of all real numbers

$\mathbb{C}$: Set of all complex numbers

(*) Open statements

- It is a sentence which involves one or more variables which turns out as a statement (Proposition) when the variables (s) take permissible values. Obviously these statements will be either true or false.

(or)

- The declarative sentences like $x+2=3$, $x=\sqrt{2}$, $x>0$, x divisible by 2, in these sentences 'x' is not defined, such statements are called as open statement and 'x' is called the free variables.

Note:-

- The open statement can be converted into propositions by giving the values to 'x' from Universal Set (U).
- Open statements are denoted by $p(x)$, $q(x)$, $r(x)$ and so on.
- Consider $x+3=6$ and the set of real number R. Now, this sentences becomes a proposition if 'x' is replaced by any element of R.
- For example, if 'x' is replaced by 3, then the open statement $x+3=6$ is a true proposition and if 'x' is replaced by 5, it becomes a false proposition. ~~Hence~~ Here we say R is a universe (or universe of discourse).

Examples

① $p(x) : x \leq 5$

Sol:- $\cdot\ p(1) : 1 < 5$; $p(2) : 2 < 5$; $p(3) : 3 < 5$; $p(4) : 4 < 5$; $p(5) : 5 = 5$ are all True statements

$\cdot\ p(6) : 6 \nless 5$; $p(7) : 7 < 5$ ---- are all False statements

② $q(x, y) : x + 5 = y$

Sol:- $q(1, 6)$; $q(-5, 0)$; $q(95, 100)$ are statements with truth value "True"(T).

$q(1, 7)$; $q(5, 9)$ ... are statement with truth value "False"(F).

③ $r(x, y, z) : x^2 + y^2 = z^2$

Sol:- $r(3, 4, 5)$; $r(6, 8, 10)$ are True statements

$r(1, 2, 3)$; $r(4, 5, 6)$ are False statements.

(*) Quantifiers

- Open sentences involving words of the form "for all" or "for some" with reference to the variable(s) involved are called Quantifiers since quantity is involved in it.

- Sentences involved with words of the type for all / for every, symbolically $\forall$ with reference to the variable(s) are called Universal quantifiers.

- Sentences involved with words of the type for some / there exists, symbolically $\exists$ are called Existential quantifiers

- Any statement (predicate) involved with either of these two quantifiers is called a quantified Statement.

- The truth value of a quantified statement (involved with $\forall$ / $\exists$) is decided as follows with reference to a predicate say $p(x)$, $x \in S$. → Universal for x

(i) $(\forall x)[p(x)]$ is True(T) only when $p(x)$ is true for every x belonging to a set S.

(ii) $(\exists x)[p(x)]$ is False(F) only when $p(x)$ is false for every x belonging to a set S.

The Negation of a quantified statement (Predicate) $p(x), x \in S$ is written as follows

$$\sim\{\forall x, [p(x)]\} \iff \exists x, [\sim p(x)]$$

$$\sim\{\exists x, [p(x)]\} \iff \forall x, [\sim p(x)]$$

Worked problems

① If $A=\{1,2,3,4,5\}$ is the Universal Set, determine the truth values of each of the following statements.

(i) $(\forall x \in A)(x+2<10)$ (ii) $(\exists x \in A)(x+2=10)$

(iii) $(\forall x \in A)(x^2 \leq 25)$ (iv) $(\exists x \in A)(x^2-5x+6=0)$

Sol:- (i) Let $p_1(x): x+2<10$

$p_1(1)=3<10$, $p_1(2)=4<10$, $p_1(3)=5<10$, $p_1(4)=6<10$, $p_1(5)=7<10$, These are all true. Equivalently, $p_1(x)$ is true for every $x \in A$.

Thus the truth value of $p_1(x)$ is True (T).

(ii) Let $p_2(x): x+2=10$

$p_2(1)=3\neq10$; $p_2(2)=4\neq10$, $p_2(3)=5\neq10$, $p_2(4)=6\neq10$, $p_2(5)=7\neq10$ These are all false. Equivalently, $p_2(x)$ is false for every $x \in A$.

Thus the truth value of $p_2(x)$ is False (F).

(iii) Let $p_3(x): x^2 \leq 25$

LHS of $p_3(x)$ for the values of A are respectively 1, 4, 9, 16, 25 which satisfy RHS: ≤ 25. $p_3(x)$ is true for every $x \in A$.

Thus the truth value of $p_3(x)$ is True (T).

(iv) Let $p_4(x) = x^2-5x+6=0$ and on solving we get $x=2,3$. These belong to A.

Thus the truth value of $p_4(x)$ is True (T).

② Write the following proposition in the symbolic form and find its negation. "If all triangles are right angled then no triangle is equiangular".

Sol:- Let S be the universal set consisting of all the triangles.

- The predicates are as follows
  - $p(x)$: x is a right angled triangle
  - $q(x)$: x is an equiangular triangle
- Symbolic form of the given statement is as follows

$$\{\forall x, [p(x)]\} \longrightarrow \{\forall x, [\sim q(x)]\} \quad \text{— ①}$$

- We have $p \longrightarrow q \Leftrightarrow \sim p \vee q$

$$\Rightarrow \sim(p \longrightarrow q) \Leftrightarrow \sim(\sim p \vee q)$$

$$\Rightarrow \sim(p \longrightarrow q) \Leftrightarrow p \wedge \sim q \quad \text{— ②}$$

using ② in ① gives the negation of the given proposition

i.e., $[\forall x, [p(x)]] \wedge [\exists x, q(x)]$

This statement in the verbal form is as follows

"All triangles are right angled and there are some triangles which are equiangular".

3. For the following statements, the universe comprises all non-zero integers. Determine the truth value of each statement.

(a) $\exists x \exists y [xy = 1]$ (b) $\exists x \forall y [xy = 1]$ (c) $\forall x \exists y [xy = 1]$

(d) $\exists x \exists y [(2x+y=5) \wedge (x-3y=-8)]$ (e) $\exists x \exists y [(3x-y=7) \wedge (2x+4y=3)]$

Sol:- Let S be the universal set containing all non-zero integers.

(a) $\exists x \exists y [xy = 1]$

This statement is true, since there exist non-zero integers $x=1$ and $y=1$ such that $xy=1$

(b) $\exists x \forall y [xy = 1]$

This statement is false, since for every non-zero integer $x=1$, $y=2$ is also a non-zero integer, but $xy=(2)(1)=2$ which is greater than ~~~~ 1. Hence $xy \neq 1$.

(c) $\forall x \exists y [xy = 1]$

This statement is false, because for $x=2$, there doesn't exist a non-zero integer y such that the statement holds for any non-zero integer y, $xy = 2x > 2.1 = 2 > 1$. Hence $xy \neq 1$.

(d) This statement just states that there exist non-zero integers x and y such that the two equations hold.

Consider, second equation $x - 3y = -8$

$$x = -8 + 3y \quad — (1)$$

Then, by inserting it in the first equation, we get

$2(-8+3y)+y=5$

$y=3$

Now it follows that $x=-8+3y=-8+3(3)=1$

Statement is true.

(v) Statement is true.

(4) Check for validity of following argument.

- If a triangle has two equal sides then it is isosceles.
- If a triangle is isosceles than it has two equal angles.
- A certain triangle ABC does not have two equal angles

∴ The triangle ABC does not have two usual sides.

Sol:- Let, $p(x)$: x has two equal sides

$q(x)$: x is isosceles

$r(x)$: x has two equal angles

a : triangle ABC

Given
$$\left.\begin{array}{r} \forall x,\ p(x) \rightarrow q(x) \\ \forall x,\ q(x) \rightarrow r(x) \\ \neg r(a) \\ \hline \therefore\ \neg p(a) \end{array}\right\} \Rightarrow \begin{array}{r} p(a) \rightarrow q(a) \\ q(a) \rightarrow r(a) \\ \neg r(a) \\ \hline \therefore\ \neg p(a) \end{array}$$
universal specification

Consider, $\{p(a) \rightarrow q(a)\} \wedge \{q(a) \rightarrow r(a)\} \wedge (\neg r(a))$

$\Rightarrow \{p(a) \rightarrow r(a)\} \wedge (\neg r(a))$ {by Rule of syllogism}

$\Rightarrow \neg p(a)$ {by Rule of Modus Tollens}

Thus, the given argument is valid

5 Consider the following open statement on set of all real numbers as universe:

$p(x): x \geq 0$, $q(x): x^2 \geq 0$, $r(x): x^2-3x-4=0$, $s(x): x^2-3 \geq 0$

Then find truth value of

(i) $\exists x, [p(x) \wedge q(x)]$

(ii) $\forall x, [p(x) \longrightarrow q(x)]$

(iii) $\forall x, [q(x) \longrightarrow s(x)]$

(iv) $\forall x, [r(x) \vee s(x)]$

Sol:- R: set of all real numbers

(i) $\exists x, [p(x) \wedge q(x)]$

We note that, there exists a real number $x=1$ for which both $p(x)$ and $q(x)$ are true

$\therefore \exists x, [p(x) \wedge q(x)]$ is a true statement

It's truth value is 1.

(ii) $\forall x, [p(x) \longrightarrow q(x)]$

For every real number x, $q(x)$ is true

$\therefore \forall x, [p(x) \longrightarrow q(x)]$ is true.

It's truth value is 1

(iii) $\forall x, [q(x) \longrightarrow s(x)]$

We note that $s(x)$ is false and $q(x)$ is true for $x=1$

Thus, $\forall x, [q(x) \longrightarrow s(x)]$ is false

It's truth value is 0

(iv) $\forall x, [r(x) \vee s(x)]$

- We note that, $r(x)$ is true only for $x=4$ and $x=-1$. whereas $r(s)$ and $s(x)$ are false for $x=1$.
- Thus $r(x) \vee s(x)$ is not always true

$\therefore \forall x, [r(x) \vee s(x)]$ is false.

It's truth value is 0.

(6) Determine the truth value of the following statements if the universe comprises of all non-zero integers

(i) $\exists x \exists y [xy=2]$ (ii) $\exists x \forall y [xy=2]$ (iii) $\forall x \exists y [xy=2]$

(iv) $\exists x \exists y [(3x+y=8) \wedge (2x-y)=7]$ (v) $\exists x \exists y [(4x+2y=3) \wedge (x-y=1)]$

S:- (*) Let 'S' be the set of all non-zero integers

(i) $\exists x \exists y [xy=2]$

- This statement is true, since there exists non-zero integers $x=2, y=1$ such that $xy=(2\times1)=2$. Therefore, the truth value is True (T) or 1.

(ii) $\exists x \forall y [xy=2]$

- This statement is false, since there exists non-zero integer $x=2$ and for all non-zero integers y, we can obtain $xy=(2)(2)=4 \neq 2$. Thus the truth value is False (F) or 0.

(iii) $\forall x \exists y [xy = 2]$

- This statement is false, since for all non-zero integer x, there exists non-zero integer $y = 2$, such that $xy = (2)(2) = 4 \neq 2$. Therefore, the truth value is False (F) or 0.

(iv) $\exists x \exists y [(3x + y = 8) \wedge (2x - y) = 7]$

Consider, $3x + y = 8 \Rightarrow y = 8 - 3x$ — (1)

Now, $2x - y = 7 \Rightarrow y = 2x - 7$ — (2)

Substitute (1) into (2), gives

$$(8 - 3x) = (2x - 7)$$

$$\Rightarrow 8 - 3x - 2x + 7 = 0$$

$$\Rightarrow -5x + 15 = 0$$

$$\Rightarrow -5x = -15$$

$$\Rightarrow \boxed{x = 3}$$

(2) $\Rightarrow y = 2(3) - 7 = 1 \Rightarrow \boxed{y = -1}$

- This statement is True, since there exists non-zero integers $x = 3$ and $y = -1$ belonging to S. Hence simplifies two equations. Therefore the truth value is True (T) or 1.

~~(v) $\exists x \exists y [(4x + 2y = 3) \wedge (x - y) = 1$~~

~~Consider $4x + 2y = 3 \Rightarrow 4x = 3 - 2y \Rightarrow x = \frac{3 - 2y}{4}$~~

~~Now, $x - y = 1 \Rightarrow x =$~~

(v) $\exists x \exists y\ [(4x+2y=3) \wedge (x-y)=1]$

Consider, $x-y=1 \Rightarrow x=1+y$ — (1)

Now, $4x+2y=3$

$\Rightarrow 4(1+y)+2y=3$ (using (1))

$\Rightarrow 4+4y+2y=3$

$\Rightarrow 6y=3-4$

$\Rightarrow 6y=-1$

$\Rightarrow \boxed{y=-\frac{1}{6}}$

$\boxed{x=1-\frac{1}{6}=\frac{5}{6}}$

This statement is False, since there doesn't exists non-zero integers $x=\frac{5}{6}$ and $y=-\frac{1}{6}$. Therefore, the truth value is False (F) or 0.

(7) If $p(x): x \geq 0$, $q(x): x^2 \geq 0$, $r(x): x^2-3x-4=0$, $s(x): x^2-3>0$

Determine the truth or falsity of the following statement:

(i) $\exists x\,[p(x) \wedge q(x)]$ (ii) $\forall x\,[p(x) \rightarrow q(x)]$ (iii) $\forall x\,[q(x) \rightarrow s(x)]$

(iv) $\forall x\,[r(x) \wedge s(x)]$ (v) $\exists x\,[p(x) \wedge r(x)]$ (vi) $\forall x\,[r(x) \rightarrow p(x)]$

(vii) $\exists x\,[r(x) \rightarrow \neg p(x)]$